Exercise 2C

1 a $(x_1, y_1) = (3, 2), r = 4$

So $(x-3)^2 + (y-2)^2 = 4^2$

$(x-3)^2 + (y-2)^2 = 16$

b $(x_1, y_1) = (-4, 5), r = 6$

So $(x-(-4))^2 + (y-5)^2 = 6^2$

$(x+4)^2 + (y-5)^2 = 36$

c $(x_1, y_1) = (5, -6), r = 2\sqrt{3}$

So $(x-5)^2 + (y-(-6))^2 = (2\sqrt{3})^2$

$(x-5)^2 + (y+6)^2 = 2^2(\sqrt{3})^2$

$(x-5)^2 + (y+6)^2 = 4 \times 3$

$(x-5)^2 + (y+6)^2 = 12$

d $(x_1, y_1) = (2a, 7a), r = 5a$

So $(x-2a)^2 + (y-7a)^2 = (5a)^2$

$(x-2a)^2 + (y-7a)^2 = 25a^2$

e $(x_1, y_1) = (-2\sqrt{2}, -3\sqrt{2}), r = 1$

So $\left(x-(-2\sqrt{2})\right)^2 + \left(y-(-3\sqrt{2})\right)^2 = 1^2$

$(x+2\sqrt{2})^2 + (y+3\sqrt{2})^2 = 1$

2 a $(x+5)^2 + (y-4)^2 = 9^2$

$(x-(-5))^2 + (y-4)^2 = 9^2$

The centre of the circle is (–5, 4) and the radius is 9.

b $(x-7)^2 + (y-1)^2 = 16$

$(x-7)^2 + (y-1)^2 = 4^2$

The centre of the circle is (7, 1) and the radius is 4.

c $(x+4)^2 + y^2 = 25$

$(x-(-4))^2 + (y-0)^2 = 5^2$

The centre of the circle is (–4, 0) and the radius is 5.

2 d $(x+4a)^2+(y+a)^2=144a^2$

$(x-(-4a))^2+(y-(-a))^2=(12a)^2$

The centre of the circle is $(-4a, -a)$ and the radius is $12a$.

e $(x-3\sqrt{5})^2+(y+\sqrt{5})^2=27$

$(x-3\sqrt{5})^2+(y-(-\sqrt{5}))^2=(\sqrt{27})^2$

Now $\sqrt{27}=\sqrt{9\times3}=\sqrt{9}\times\sqrt{3}=3\sqrt{3}$

The centre of the circle is $(3\sqrt{5}, -\sqrt{5})$ and the radius is $3\sqrt{3}$.

3 a Substitute $x = 4$, $y = 8$ into $(x - 2)^2 + (y - 5)^2 = 13$

$(x - 2)^2 + (y - 5)^2 = (4 - 2)^2 + (8 - 5)^2 = 2^2 + 3^2 = 4 + 9 = 13$ ✓

So the circle passes through (4, 8).

b Substitute $x = 0$, $y = -2$ into $(x + 7)^2 + (y - 2)^2 = 65$

$(x +7)^2 + (y - 2)^2 = (0 + 7)^2 + (-2 - 2)^2 = 7^2 + (-4)^2 = 49 + 16 = 65$ ✓

So the circle passes through (0, –2).

c Substitute $x = 7$, $y = -24$ into $x^2 + y^2 = 25^2$

$x^2 + y^2 = 7^2 + (-24)^2 = 49 + 576 = 625 = 25^2$ ✓

So the circle passes through (7, –24).

d Substitute $x = 6a$, $y = -3a$ into $(x - 2a)^2 + (y + 5a)^2 = 20a^2$

$(x - 2a)^2 + (y + 5a)^2 = (6a - 2a)^2 + (-3a + 5a)^2 = (4a)^2 + (2a)^2 = 16a^2 + 4a^2 = 20a^2$ ✓

So the circle passes through $(6a, -3a)$.

e Substitute $x=\sqrt{5}, y=-\sqrt{5}$ into $(x-3\sqrt{5})^2+(y-\sqrt{5})^2=(2\sqrt{10})^2$

$(x-3\sqrt{5})^2+(y-\sqrt{5})^2=(\sqrt{5}-3\sqrt{5})^2+(-\sqrt{5}-\sqrt{5})^2=(-2\sqrt{5})^2+(-2\sqrt{5})^2$

$=4\times5+4\times5=20+20=40=(\sqrt{40})^2$

Now $\sqrt{40}=\sqrt{4\times10}=\sqrt{4}\times\sqrt{10}=2\sqrt{10}$ ✓

So the circle passes through $(\sqrt{5}, -\sqrt{5})$.

4 The radius of the circle is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-8)^2+((-2)-1)^2}$$
$$=\sqrt{4^2+3^2}$$
$$=\sqrt{16+9}$$
$$=\sqrt{25}$$
$$=5$$

The centre of the circle is (8, 1) and the radius is 5.

So $(x - 8)^2 + (y - 1)^2 = 5^2$

or $(x - 8)^2 + (y - 1)^2 = 25$

5 P (5, 6) and Q (−2, 2)

The centre of the circle is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{5+(-2)}{2},\frac{6+2}{2}\right)=\left(\frac{3}{2},\frac{8}{2}\right)=\left(\frac{3}{2},4\right)$$

The radius of the circle is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{\left(5-\frac{3}{2}\right)^2+(6-4)^2}$$
$$=\sqrt{\left(\frac{7}{2}\right)^2+(2)^2}$$
$$=\sqrt{\frac{49}{4}+4}$$
$$=\sqrt{\frac{49}{4}+\frac{16}{4}}$$
$$=\sqrt{\frac{65}{4}}$$

So the equation of the circle is

$$\left(x-\frac{3}{2}\right)^2+(y-4)^2=\left(\sqrt{\frac{65}{4}}\right)^2 \text{ or } \left(x-\frac{3}{2}\right)^2+(y-4)^2=\frac{65}{4}$$

6 Substitute $x=1,\ y=-3$ into $(x-3)^2+(y+4)^2=r^2$

$$(1-3)^2+(-3+4)^2=r^2$$
$$(-2)^2+(1)^2=r^2$$
$$5=r^2$$
So $r=\sqrt{5}$

7 **a** Substitute $(2,2)$ into $(x-2)^2+(y-4)^2=r^2$

$$(2-2)^2+(2-4)^2=r^2$$
$$0^2+(-2)^2=r^2$$
$$r^2=4$$
$$r=2$$

7 b The distance between (2, 2) and $\left(2+\sqrt{3},5\right)$ is

$$\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}=\sqrt{\left(2+\sqrt{3}-2\right)^2+\left(5-2\right)^2}$$
$$=\sqrt{\left(\sqrt{3}\right)^2+3^2}$$
$$=\sqrt{3+9}$$
$$=\sqrt{12}$$

The distance between (2, 2) and $\left(2-\sqrt{3},5\right)$ is

$$\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}=\sqrt{\left(2-\sqrt{3}-2\right)^2+\left(5-2\right)^2}$$
$$=\sqrt{\left(-\sqrt{3}\right)^2+\left(3\right)^2}$$
$$=\sqrt{3+9}$$
$$=\sqrt{12}$$

The distance between $\left(2+\sqrt{3},5\right)$ and $\left(2-\sqrt{3},5\right)$ is

$$\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}=\sqrt{\left(\left(2-\sqrt{3}\right)-\left(2+\sqrt{3}\right)\right)^2+\left(5-5\right)^2}$$
$$=\sqrt{\left(2-\sqrt{3}-2-\sqrt{3}\right)^2+0^2}$$
$$=\sqrt{\left(-2\sqrt{3}\right)^2}$$
$$=\sqrt{4\times 3}$$
$$=\sqrt{12}$$

PQ, *QR* and *PR* all equal $\sqrt{12}$.

So ΔPQR is equilateral.

8 a Rearrange $x^2+y^2-4x-11=0$ into the form $\left(x-a\right)^2+y^2=r^2$

$x^2-4x-11+y^2=0$

Completing the square gives

$\left(x-2\right)^2-4-11+y^2=0$

$\left(x-2\right)^2+y^2=15$

b Centre of the circle = (2, 0), radius = $\sqrt{15}$

9 a $x^2+y^2-10x+4y-20=0$

$x^2-10x+y^2+4y-20=0$

Completing the square gives

$(x-5)^2-25+(y+2)^2-4-20=0$

$(x-5)^2+(y+2)^2=49$

9 b Centre of the circle = (5, −2), radius = 7

10 a $x^2 + y^2 - 2x + 8y - 8 = 0$
$x^2 - 2x + y^2 + 8y - 8 = 0$
Completing the square gives
$(x-1)^2 - 1 + (y+4)^2 - 16 - 8 = 0$
$(x-1)^2 + (y+4)^2 = 25$
Centre of the circle = (1, −4), radius = 5

b $x^2 + y^2 + 12x - 4y = 9$
$x^2 + 12x + y^2 - 4y = 9$
Completing the square gives
$(x+6)^2 - 36 + (y-2)^2 - 4 = 9$
$(x+6)^2 + (y-2)^2 = 49$
Centre of the circle = (−6, 2), radius = 7

c $x^2 + y^2 - 6y = 22x - 40$
$x^2 - 22x + y^2 - 6y = -40$
Completing the square gives
$(x-11)^2 - 121 + (y-3)^2 - 9 = -40$
$(x-11)^2 + (y-3)^2 = 90$
Centre of the circle = (11, 3), radius = $\sqrt{90} = 3\sqrt{10}$

d $x^2 + y^2 + 5x - y + 4 = 2y + 8$
$x^2 + 5x + y^2 - 3y = 4$
Completing the square gives
$$\left(x+\frac{5}{2}\right)^2 - \frac{25}{4} + \left(y-\frac{3}{2}\right)^2 - \frac{9}{4} = 4$$
$$\left(x+\frac{5}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{25}{2}$$
Centre of the circle = $\left(-\frac{5}{2}, \frac{3}{2}\right)$, radius = $\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

e $2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$
$2x^2 - 8x + 2y^2 + 8y = -3$
Completing the square gives
$2((x-2)^2 - 4) + 2((y+2)^2 - 4) = -3$
$2(x-2)^2 - 8 + 2(y+2)^2 - 8 = -3$
$2(x-2)^2 + 2(y+2)^2 = 13$
$(x-2)^2 + (y+2)^2 = \frac{13}{2}$
Centre of the circle = (2, −2), radius = $\sqrt{\frac{13}{2}} = \frac{\sqrt{26}}{2}$

11 a $x^2 + y^2 + 12x + 2y = k$
$x^2 + 12x + y^2 + 2y = k$
$(x+6)^2 - 36 + (y+1)^2 - 1 = k$
$(x+6)^2 + (y+1)^2 = k + 37$
Centre of the circle = (−6, −1)

11 b A circle must have a positive radius, so $k + 37 > 0$
So $k > -37$

12 Find the center of the circle by completing the square
$(x + 3)^2 - 9 + (y - 7)^2 - 49 = 483$
$(x + 3)^2 + (y - 7)^2 = 541$
Centre of the circle = (-3,7)
Q is twice the distance past the center from P(7,14),
so Q = (-13,28)

13 (x − k)2 + y2 = 41, (3, 4)

Substitute $x = 3$ and $y = 4$ into the equation $(x - k)^2 + y^2 = 41$
$(3 - k)^2 + 4^2 = 41$
$k^2 - 6k + 9 + 16 = 41$
$k^2 - 6k - 16 = 0$
$(k + 2)(k - 8) = 0$
$k = -2$ or $k = 8$

Challenge

1 $(x - k)^2 + (y - 2)^2 = 50$, $(4, -5)$
Substitute $x = 4$ and $y = -5$ into the equation $(x - k)^2 + (y - 2)^2 = 50$
$(4 - k)^2 + (-5 - 2)^2 = 50$
$k^2 - 8k + 16 + 49 = 50$
$k^2 - 8k + 15 = 0$
$(k - 3)(k - 5) = 0$
$k = 3$ or $k = 5$
$(x - 3)^2 + (y - 2)^2 = 50$ or $(x - 5)^2 + (y - 2)^2 = 50$

2 $x^2 + y^2 + 2fx + 2gy + c = 0$
Rearranging the equation:
$x^2 + 2fx + y^2 + 2gy + c = 0$
Completing the square gives
$(x + f)^2 - f^2 + (y + g)^2 - g^2 + c = 0$
$(x + f)^2 + (y + g)^2 = f^2 + g^2 - c$
The centre of the circle is $(-f, -g)$ and the radius is $\sqrt{f^2 + g^2 - c}$.